

MONOTONICITY AND SYMMETRY OF NONNEGATIVE SOLUTIONS TO

$$-\Delta u = f(u)$$

IN HALF-PLANES AND STRIPS

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A Ireneo con profonda stima e amicizia.

ABSTRACT. We consider nonnegative solutions to $-\Delta u = f(u)$ in half-planes and strips, under zero Dirichlet boundary condition. Exploiting a rotating&sliding line technique, we prove symmetry and monotonicity properties of the solutions, under very general assumptions on the nonlinearity f . In fact we provide a unified approach that works in all the cases $f(0) < 0$, $f(0) = 0$ or $f(0) > 0$. Furthermore we make the effort to deal with nonlinearities f that may be not locally-Lipschitz continuous. We also provide explicit examples showing the sharpness of our assumptions on the nonlinear function f .

1. INTRODUCTION AND MAIN RESULTS

We consider the problem of classifying solutions to

$$(1.1) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

under very general assumptions on the nonlinear function f . Here, by \mathbb{R}_+^2 , we mean the open half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$.

Let us recall that, in our two previous works [9, 16], we proved the following result.

Theorem 1.1. *Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to*

$$(1.2) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

with f locally Lipschitz continuous on $[0, +\infty)$. Then

AF is partially supported by ERC-2011-grant: *Epsilon* and by ERC-2013-grant: *COMPAT*. BS is partially supported by the Italian PRIN Research Project 2007: *Metodi Variazionali e Topologici nello Studio di Fenomeni non Lineari*, and is also partially supported by ERC-2011-grant: *Epsilon*.

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2010 Mathematics Subject Classification: 35J61, 35B51, 35B06.

- i) if $f(0) \geq 0$, either u vanishes identically, or u is positive on \mathbb{R}_+^2 with $\frac{\partial u}{\partial y} > 0$ in \mathbb{R}_+^2 .
- ii) if $f(0) < 0$, either u is positive on \mathbb{R}_+^2 , with $\frac{\partial u}{\partial y} > 0$ in \mathbb{R}_+^2 , or u is one-dimensional and periodic (and unique).

The case $f(0) \geq 0$ was first treated in [9] (in which also the case of the p -Laplace operator is considered) while the case $f(0) < 0$ was carried out later, in the paper [16]. Both of them are based on a refined version of the *moving plane method* [26] (see also [5, 22]). More precisely, the authors of [9] exploit a *rotating line technique* and a *sliding line technique*, while in [16] we have used a refinement of these techniques combined with the *unique continuation principle*, needed to handle the new, challenging and difficult case of nonnegative solutions. The techniques developed in [9, 16] also provided an affirmative answer to a conjecture and to an open question posed by Berestycki, Caffarelli and Nirenberg in [1, 2].

In this paper we first give a (new) unique/unified proof to Theorem 1.1 and at the same time we make the effort to deal with the case of continuous nonlinearities f that fulfills very weak and general regularity assumptions, i.e., less regular than locally Lipschitz continuous. See assumptions (h_1) – (h_4) in Section 2.

In this direction we have the following results.

Theorem 1.2. *Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to*

$$(1.3) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^2, \\ u > 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

and assume that either

- i) $f(0) = 0$ and f fulfills (h_1) , (h_2) and (h_3) ,

or

- ii) $f(0) \neq 0$ and f fulfills (h_1) , (h_2) and (h_4) .

Then

$$\frac{\partial u}{\partial y} > 0 \quad \text{in } \mathbb{R}_+^2.$$

We observe that the conclusion of Theorem 1.2 is not true if we drop the assumption (h_3) in item i). This will be discussed in the last section of the paper.

Let us point out that, with the same technique, we can also prove a symmetry and monotonicity result in strips, for possibly unbounded solutions. More precisely, with the notation $\Sigma_{2b} := \{(x, y) \in \mathbb{R}^2 : y \in (0, 2b)\}$, $b > 0$, we have the following:

Theorem 1.3. *Let $u \in C^2(\overline{\Sigma_c})$ for any $c < 2b$ be a solution to*

$$(1.4) \quad \begin{cases} -\Delta u = f(u), & \text{in } \Sigma_{2b} \\ u > 0, & \text{in } \Sigma_{2b} \\ u = 0, & \text{on } \{y = 0\} \end{cases}$$

and assume that either

i) $f(0) = 0$ and f fulfills (h_1) , (h_2) and (h_3) ,

or

ii) $f(0) \neq 0$ and f fulfills (h_1) , (h_2) and (h_4) .

Then

$$\frac{\partial u}{\partial y} > 0 \quad \text{in} \quad \Sigma_b.$$

If $u \in C^2(\overline{\Sigma_{2b}})$ and $u = 0$ on $\partial\Sigma_{2b}$, then u is symmetric about $\{y = b\}$.

As already observed for Theorem 1.2, the conclusion of Theorem 1.3 is not true if we drop the assumption (h_3) in item i) (see section 6 of this paper).

The Theorem above complements Theorem 1.3 of [16] and also provides an affirmative answer to an (extended version of an) open question posed by Berestycki, Caffarelli and Nirenberg in [1].

Next we prove a symmetry result in the case of the half-plane.

Theorem 1.4. *Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to*

$$(1.5) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

with f locally Lipschitz continuous on $[0, +\infty)$ and satisfying

$$(1.6) \quad \exists p > 1 \quad : \quad \lim_{t \rightarrow +\infty} \frac{f(t)}{t^p} = l \in (0, +\infty).$$

Then u is bounded and one-dimensional, i.e.

$$u(x, y) = u_0(y) \quad \forall (x, y) \in \mathbb{R}_+^2,$$

for some bounded function $u_0 \in C^2([0, +\infty))$.

As a consequence of the results above we also obtain the following

Corollary 1.5. *Let f be locally Lipschitz continuous on $[0, +\infty)$ and satisfying*

$$(1.7) \quad f(t) > 0 \quad \forall t > 0,$$

$$(1.8) \quad \exists p > 1 \quad : \quad \lim_{t \rightarrow +\infty} \frac{f(t)}{t^p} = l \in (0, +\infty).$$

Then

i) if $f(0) = 0$, the only solution of class $C^2(\overline{\mathbb{R}_+^2})$ of

$$(1.9) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

is $u \equiv 0$,

ii) if $f(0) > 0$, problem (1.9) has no solution of class $C^2(\overline{\mathbb{R}_+^2})$.

We conclude this section with the following classification result

Theorem 1.6. *Let f be non decreasing locally Lipschitz continuous on $[0, +\infty)$ satisfying*

$$(1.10) \quad f(0) \geq 0$$

Then, the problem

$$(1.11) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

has a nontrivial solution of class $C^2(\overline{\mathbb{R}_+^2})$ if and only if $f \equiv 0$. In the latter case u is necessarily linear, i.e., $u(x, y) = cy$, for some constant $c > 0$.

Furthermore, when $f \not\equiv 0$, we have that

- i) *if $f(0) = 0$, the only solution of class $C^2(\overline{\mathbb{R}_+^2})$ of (1.11) is $u \equiv 0$.*
- ii) *if $f(0) > 0$, problem (1.11) has no solution of class $C^2(\overline{\mathbb{R}_+^2})$.*

Remark 1.7. *The assumption (1.10) is sharp. Indeed, the function $u(x, y) = 1 - \cos y$ is a nontrivial solution of*

$$(1.12) \quad \begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

and $f(0) = -1 < 0$.

Remark 1.8.

- i) *Theorem 1.6 applies, for instance, to the functions $f(u) = u^p + c$, with $p \geq 1$ and $c \geq 0$. In particular, for $c = 0$ (i.e. for $f(u) = u^p$) we obtain a new and different proof of a celebrated result of Gidas and Spruck [21] (see also [9]).*
- ii) *Note also that one can apply item i) of Corollary 1.5 to $f(u) = u^p$, $p > 1$, to obtain another new and different proof of the above mentioned result of Gidas and Spruck [21].*

In this work we focused on the two-dimensional case and we provided a precise description of the situation under very general assumptions (both on f and on u). Differently from the two-dimensional case, the situation is not yet well-understood for dimensions $N \geq 3$. For results in the higher dimensional case (and with additional assumptions on f and on u) we refer to [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

2. ASSUMPTIONS AND PRELIMINARY RESULTS

We start discussing the main assumptions on the nonlinearity f . In Theorem 1.1 we assume that f is locally Lipschitz continuous on $[0, +\infty)$. This allow us to deal with nonnegative solutions. In Theorem 1.2 we restrict our attention to positive solutions weakening the assumptions on the nonlinearity. It is convenient to set the following

(h_1) f is continuous in $[0, +\infty)$.

(h_2) f is *locally Lipschitz continuous from above* in $[0, \infty)$, i.e., for every $b > 0$ there is $L_b > 0$:

$$f(u) - f(v) \leq L_b(u - v) \quad \text{for any } 0 \leq v \leq u \leq b.$$

(h_3) For any $\bar{t} \in [0, +\infty)$, there exists $\delta = \delta(\bar{t}) > 0$ such that

$$(2.1) \quad f(s) - f(t) \leq g(t - s) \quad \forall s \leq t \in [\bar{t} - \frac{\delta}{2}, \bar{t} + \frac{\delta}{2}] \cap [0, +\infty)$$

with $g \in C^0[0, \delta]$, $g(0) = 0$ and, either g vanishes identically in $[0, \delta]$, or it is positive and non decreasing in $[0, \delta]$ with

$$(2.2) \quad \int_0^\delta \frac{ds}{\sqrt{G(s)}} = \infty$$

where $G(s) := \int_0^s g(t) dt$.

(h_4) Condition (h_3) holds but only for any $\bar{t} \in (0, +\infty)$.

Note that the hypothesis (h_2) is very weak (actually it is not even enough to ensure the continuity of f) and it is clearly satisfied by any non increasing function. Actually, all the nonlinearities of the form

$$f(s) := f_1(s) + f_2(s),$$

for some non increasing continuous function $f_1(\cdot)$ in $[0, \infty)$ and some $f_2(\cdot)$, which is locally Lipschitz continuous in $[0, \infty)$, satisfy both (h_1) and (h_2).

We also observe that the assumption (h_3) is natural. Indeed, the conditions imposed on g are the well-known optimal assumptions ensuring the validity of the strong maximum principle and the Hopf's lemma (see [25] for instance). It is also clear that every locally Lipschitz continuous function on $[0, \infty)$ satisfies (h_1), (h_2) and (h_3) with $g(t) = L_\delta t$, $L_\delta > 0$ being any constant larger than the Lipschitz constant of f on the interval $[0, \delta]$. On the other hand, the converse is not true. This is the case e.g. when $f(\cdot)$ has the form

$$f(s) := g(s) + c,$$

where $g(s) \equiv s \log(s)$ in some interval $(0, \delta)$, $\delta > 0$ and $g(\cdot)$ of class C^1 in $(0, \infty)$. It is easy to verify that such a nonlinearity fulfills (h_3) but it is not Lipschitz continuous at zero.

Now we are ready to prove

Proposition 2.1 (Weak Comparison Principle in domains of small measure). *Let $N \geq 1$, assume that f fulfills (h_2) and fix a real number $k > 0$. Then there exists $\vartheta = \vartheta(N, k, f) > 0$ such that, for any domain $D \subset \mathbb{R}^N$, with $\mathcal{L}(D) \leq \vartheta$, and any $u, v \in H^1(D) \cap C^0(\overline{D})$ such that*

$$(2.3) \quad \begin{cases} -\Delta u - f(u) \leq -\Delta v - f(v) & \text{in } D, \\ 0 \leq u, v \leq k & \text{in } D, \\ u \leq v & \text{on } \partial D, \end{cases}$$

then

$$u \leq v \quad \text{in } D.$$

Proof. We use $(u - v)^+ \in H_0^1(D)$ as test function in the weak formulation of (2.3) and get

$$\begin{aligned} \int_D |\nabla(u - v)^+|^2 dx &\leq \int_D (f(u) - f(v))(u - v)^+ dx \\ &\leq L_k \int_D ((u - v)^+)^2 dx \end{aligned}$$

where L_k is the positive constant appearing in (h_2) and corresponding to $b = k > 0$. Note that L_k depends only on k and f .

An application of Poincaré inequality gives

$$\int_D |\nabla(u - v)^+|^2 dx \leq L_k (C_N (\mathcal{L}(D'))^{\frac{2}{N}}) \int_D |\nabla(u - v)^+|^2 dx,$$

where $C_N > 0$ is a constant depending only on the euclidean dimension N .

The desired conclusion then follows by choosing $\vartheta = \frac{1}{2L_k^{\frac{N}{2}} C_N}$. Indeed, the latter implies that $L_k (C_N (\mathcal{L}(D'))^{\frac{2}{N}}) < 1$ and so we get that $(u - v)^+ \equiv 0$ and the thesis. \square

Now we focus on the two-dimensional case and fix some notations.

Given $x_0 \in \mathbb{R}$, $s > 0$ and $\theta \in (0, \frac{\pi}{2})$, let $L_{x_0, s, \theta}$ be the line, with slope $\tan(\theta)$, passing through (x_0, s) . Also, let V_θ be the vector orthogonal to $L_{x_0, s, \theta}$ such that $(V_\theta, e_2) > 0$ and $\|V_\theta\| = 1$.

We denote by

$$(2.4) \quad \mathcal{T}_{x_0, s, \theta},$$

the (open) triangle delimited by $L_{x_0, s, \theta}$, $\{y = 0\}$ and $\{x = x_0\}$, and we define

$$u_{x_0, s, \theta}(x) = u(T_{x_0, s, \theta}(x)), \quad x \in \mathcal{T}_{x_0, s, \theta}$$

where $T_{x_0, s, \theta}(x)$ is the point symmetric to x , w.r.t. $L_{x_0, s, \theta}$. It is immediate to see that $u_{x_0, s, \theta}$ still fulfills $-\Delta u_{x_0, s, \theta} = f(u_{x_0, s, \theta})$ on the triangle $\mathcal{T}_{x_0, s, \theta}$.

We also consider

$$(2.5) \quad w_{x_0, s, \theta} = u - u_{x_0, s, \theta} \quad \text{on} \quad \mathcal{T}_{x_0, s, \theta}$$

and observe that

$$(2.6) \quad w_{x_0, s, \theta} \leq 0 \quad \text{on} \quad \mathcal{T}_{x_0, s, \theta} \implies \text{either } w_{x_0, s, \theta} \equiv 0 \quad \text{or} \quad w_{x_0, s, \theta} < 0 \quad \text{on} \quad \mathcal{T}_{x_0, s, \theta}$$

thanks to the assumption (h_3) (resp. to (h_4) , if u is supposed to be positive). Indeed, if $\bar{x} \in \mathcal{T}_{x_0, s, \theta}$ is such that $w_{x_0, s, \theta}(\bar{x}) = 0$ then, by the continuity of u and of $u_{x_0, s, \theta}$ we can find an open ball centered at \bar{x} , say $B_{\bar{x}} \subset \mathcal{T}_{x_0, s, \theta}$, such that

$$\begin{aligned} \forall x \in B_{\bar{x}} \quad (u(\bar{x}) - \frac{\delta}{2})^+ &\leq u(x) \leq u(\bar{x}) + \frac{\delta}{2}, \\ \forall x \in B_{\bar{x}} \quad (u(\bar{x}) - \frac{\delta}{2})^+ &\leq u_{x_0, s, \theta}(x) \leq u(\bar{x}) + \frac{\delta}{2}, \end{aligned}$$

where $\bar{t} = u(\bar{x}) = u_{x_0,s,\theta}(\bar{x})$ and $\delta = \delta(\bar{t}) > 0$ is the one provided by the assumption (h_3) (resp. (h_4)). Now, since $w_{x_0,s,\theta} \leq 0$ on $\mathcal{T}_{x_0,s,\theta}$, we can apply (2.1) to get

$$(2.7) \quad \begin{cases} \Delta(-w_{x_0,s,\theta}) \leq g(-w_{x_0,s,\theta}) & \text{in } B_{\bar{x}}, \\ -w_{x_0,s,\theta} \geq 0 & \text{in } B_{\bar{x}}, \\ -w_{x_0,s,\theta}(\bar{x}) = 0 \end{cases}$$

and the strong maximum principle (see [25] for instance) yields $w_{x_0,s,\theta} \equiv 0$ on $B_{\bar{x}}$. After that, a standard connectedness argument provides $w_{x_0,s,\theta} \equiv 0$ on the entire triangle $\mathcal{T}_{x_0,s,\theta}$.

In what follows we shall make repeated use of a refined version of the *moving plane technique* [26] (see also [5, 22]). Actually we will exploit a *rotating plane technique* and a *sliding plane technique* developed in [9, 16].

Let us give the following definition

Definition 2.2. *Given x_0, s and θ as above, we say that the condition $(\mathcal{HT}_{x_0,s,\theta})$ holds in the triangle $\mathcal{T}_{x_0,s,\theta}$ if*

$$\begin{aligned} w_{x_0,s,\theta} &< 0 \quad \text{in } \mathcal{T}_{x_0,s,\theta}, \\ w_{x_0,s,\theta} &\leq 0 \quad \text{on } \partial(\mathcal{T}_{x_0,s,\theta}) \quad \text{and} \\ w_{x_0,s,\theta} &\text{ is not identically zero on } \partial(\mathcal{T}_{x_0,s,\theta}), \end{aligned}$$

with $w_{x_0,s,\theta}$ defined in (2.5).

We have the following

Lemma 2.3 (Small Perturbations). *Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a nonnegative solution to (1.1) and assume that f fulfills (h_2) and (h_3) . Let (x_0, s, θ) and $\mathcal{T}_{x_0,s,\theta}$ be as above and assume that $(\mathcal{HT}_{x_0,s,\theta})$ holds. Then there exists $\bar{\mu} = \bar{\mu}(x_0, s, \theta) > 0$ such that*

$$(2.8) \quad \begin{cases} |\theta - \theta'| + |s - s'| < \bar{\mu}, \\ w_{x_0,s',\theta'} \leq 0 \quad \text{on } \partial(\mathcal{T}_{x_0,s',\theta'}), \\ w_{x_0,s',\theta'} \text{ is not identically zero on } \partial(\mathcal{T}_{x_0,s',\theta'}) \end{cases} \implies (\mathcal{HT}_{x_0,s',\theta'}) \text{ holds.}$$

If u is positive, the same result holds assuming only (h_4) instead of (h_3) .

Proof. Let $R > 0$ and $\tilde{\mu}$ be fixed so that

$$\bigcup_{|\theta - \theta'| + |s - s'| < \tilde{\mu}} \mathcal{T}_{x_0,s',\theta'} \cup T_{x_0,s',\theta'}(\mathcal{T}_{x_0,s',\theta'}) \subset B_R(x_0).$$

Then we set

$$k := \max_{B_R^+(x_0)} u \quad \text{and} \quad \vartheta = \vartheta(N, k, f)$$

where $\vartheta(N, k, f)$ is the one appearing in Proposition 2.1 and $B_R^+(x_0) = B_R(x_0) \cap \{y > 0\}$.

Now we fix $0 < \hat{\mu} \leq \tilde{\mu}$ such that

$$(2.9) \quad \mathcal{L}(\mathcal{T}_{x_0,s+\hat{\mu},\theta-\hat{\mu}} \setminus \mathcal{T}_{x_0,s-\hat{\mu},\theta+\hat{\mu}}) < \frac{\vartheta}{2}$$

and note that

$$\mathcal{T}_{x_0, s-\hat{\mu}, \theta+\hat{\mu}} \subset \bigcup_{|\theta-\theta'|+|s-s'|<\hat{\mu}} \mathcal{T}_{x_0, s', \theta'} \subset \mathcal{T}_{x_0, s+\hat{\mu}, \theta-\hat{\mu}}.$$

Now we can consider a compact set $K \subset \mathcal{T}_{x_0, s-\hat{\mu}, \theta+\hat{\mu}}$ so that

$$(2.10) \quad \mathcal{L}(\mathcal{T}_{x_0, s-\hat{\mu}, \theta+\hat{\mu}} \setminus K) < \frac{\vartheta}{2}.$$

By assumption we know that $w_{x_0, s, \theta} < 0$ in $\mathcal{T}_{x_0, s, \theta}$ and consequently in the compact set K . Therefore, by a uniform continuity argument, for some

$$0 < \bar{\mu} \leq \hat{\mu},$$

we can assume that

$$(2.11) \quad w_{x_0, s', \theta'} < 0 \quad \text{in } K \quad \text{for } |\theta - \theta'| + |s - s'| < \bar{\mu}.$$

By (2.9) and (2.10) we deduce that

$$\mathcal{L}(\mathcal{T}_{x_0, s', \theta'} \setminus K) < \theta$$

and, observing that $w_{x_0, s', \theta'} \leq 0$ on $\partial(\mathcal{T}_{x_0, s', \theta'} \setminus K)$ (see (2.11)), we can apply Proposition 2.1 to get that

$$w_{x_0, s', \theta'} \leq 0 \quad \text{in } \mathcal{T}_{x_0, s', \theta'} \setminus K$$

and therefore in the triangle $\mathcal{T}_{x_0, s', \theta'}$. The desired conclusion

$$w_{x_0, s', \theta'} < 0 \quad \text{in } \mathcal{T}_{x_0, s', \theta'}$$

then follows from (2.6). □

Now, from small perturbations, we move to larger translations and rotations. We have the following

Lemma 2.4 (The sliding-rotating technique). *Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a nonnegative solution to (1.1) and assume that f fulfills (h_2) and (h_3) . Let (x_0, s, θ) be as above and assume that $(\mathcal{HT}_{x_0, s, \theta})$ holds. Let $(\hat{s}, \hat{\theta})$ be fixed and assume that there exists a continuous function $g(t) = (s(t), \theta(t)) : [0, 1] \rightarrow (0, +\infty) \times (0, \frac{\pi}{2})$, such that $g(0) = (s, \theta)$ and $g(1) = (\hat{s}, \hat{\theta})$. Assume that*

$$(2.12) \quad w_{x_0, s(t), \theta(t)} \leq 0 \quad \text{on } \partial(\mathcal{T}_{x_0, s(t), \theta(t)}) \quad \text{for every } t \in [0, 1]$$

and that $w_{x_0, s(t), \theta(t)}$ is not identically zero on $\partial(\mathcal{T}_{x_0, s(t), \theta(t)})$ for every $t \in [0, 1]$.

Then

$$(\mathcal{HT}_{x_0, \hat{s}, \hat{\theta}}) \text{ holds.}$$

If u is positive, the same result holds assuming only (h_4) instead of (h_3) .

Proof. By the assumptions and exploiting Lemma 2.3 we obtain the existence of $\tilde{t} > 0$ small such that, for $0 \leq t \leq \tilde{t}$, $(\mathcal{HT}_{x_0, s(t), \theta(t)})$ holds.

We now set

$$\overline{T} \equiv \{\tilde{t} \in [0, 1] \text{ s.t. } (\mathcal{HT}_{x_0, s(t), \theta(t)}) \text{ holds for any } 0 \leq t \leq \tilde{t}\}$$

and

$$\bar{t} = \sup \overline{T}.$$

We claim that actually $\bar{t} = 1$. To prove this, assume $\bar{t} < 1$ and note that in this case we have

$$\begin{aligned} w_{x_0, s(\bar{t}), \theta(\bar{t})} &\leq 0 \quad \text{in } \mathcal{T}_{x_0, s(\bar{t}), \theta(\bar{t})} \\ w_{x_0, s(\bar{t}), \theta(\bar{t})} &\leq 0 \quad \text{on } \partial(\mathcal{T}_{x_0, s(\bar{t}), \theta(\bar{t})}) \end{aligned}$$

by continuity, and that $w_{x_0, s(\bar{t}), \theta(\bar{t})}$ is not identically zero on $\partial(\mathcal{T}_{x_0, s(\bar{t}), \theta(\bar{t})})$ by assumption.

Hence, by (2.6), we see that

$$w_{x_0, s(\bar{t}), \theta(\bar{t})} < 0 \quad \text{in } \mathcal{T}_{x_0, s(\bar{t}), \theta(\bar{t})}.$$

Therefore $(\mathcal{HT}_{x_0, s(\bar{t}), \theta(\bar{t})})$ holds and using once again Lemma 2.3, we can find a sufficiently small $\varepsilon > 0$ so that $(\mathcal{HT}_{x_0, s(t), \theta(t)})$ holds for any $0 \leq t \leq \bar{t} + \varepsilon$, which contradicts the definition of \bar{t} . \square

3. FURTHER PRELIMINARY RESULTS

In this section we assume that u is a nonnegative and non trivial solution of (1.1), i.e., $u \not\equiv 0$. We also suppose that f fulfills (h_1) and, when $f(0) = 0$, also that f fulfills (h_3) .

Given $x_0 \in \mathbb{R}$, let us set

$$(3.1) \quad B_r^+(x_0) = B_r(x_0) \cap \{y > 0\}$$

where $B_r(x_0)$ denotes the two-dimensional open ball centered at $(x_0, 0)$ and of radius $r > 0$.

We claim that, for some $\bar{r} > 0$ and for some $\bar{\theta} = \bar{\theta}(\bar{r}) \in (0, \frac{\pi}{2})$,

$$(3.2) \quad \frac{\partial u}{\partial V_\theta} > 0 \quad \text{in } B_{\bar{r}}^+(x_0) \quad \text{for } -\bar{\theta} \leq \theta \leq \bar{\theta}.$$

Case 1: $f(0) < 0$.

Since $\partial_{xx}u(x_0, 0) = 0$, we have that

$$-\partial_{yy}u(x_0, 0) = -\Delta u(x_0, 0) = f(u(x_0, 0)) = f(0) < 0.$$

Recalling that $u \in C^2(\overline{\mathbb{R}_+^2})$, we conclude that we can take $\bar{r} > 0$ small such that

$$\partial_{yy}u > 0 \quad \text{in } B_{\bar{r}}^+(x_0).$$

Exploiting again the fact that $u \in C^2(\overline{\mathbb{R}_+^2})$, we can consequently deduce that

$$(3.3) \quad \frac{\partial}{\partial V_\theta} \left(\frac{\partial u}{\partial V_\theta} \right) > 0 \quad \text{in } B_{\bar{r}}^+(x_0) \quad \text{for } -\bar{\theta} \leq \theta \leq \bar{\theta}.$$

Also, since we assumed that u is nonnegative in \mathbb{R}_+^2 , it follows that

$$(3.4) \quad \frac{\partial u}{\partial V_\theta}(x, 0) \geq 0 \quad \text{for any } -\bar{\theta} \leq \theta \leq \bar{\theta} \quad \text{and for any } x \in \mathbb{R}.$$

Combining (3.3) and (3.4), we deduce (3.2).

Case 2: $f(0) \geq 0$ and $u \not\equiv 0$.

In this case we first observe that

$$(3.5) \quad \begin{cases} u > 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial u}{\partial y}(x, 0) > 0 & \forall x \in \mathbb{R}. \end{cases}$$

Indeed, if $f(0) > 0$ and $u(\bar{x}) = 0$ with $\bar{x} \in \mathbb{R}_+^2$, then $\Delta u \leq 0$ in an open connected neighbourhood of \bar{x} by the continuity of u and f . The classical maximum principle and a standard connectedness argument imply that $u \equiv 0$ on \mathbb{R}_+^2 . The latter contradicts the non triviality of u . Therefore $u > 0$ everywhere and the classical Hopf lemma provides the second claim in (3.5).

To treat the case $f(0) = 0$ we follow the arguments leading to (2.7) and (2.6). More precisely, when $f(0) = 0$ and $u(\bar{x}) = 0$, the continuity of u and (2.1) with $\bar{t} = 0$ tell us that u satisfies the inequality $\Delta u = -f(u) = f(0) - f(u) \leq g(u)$ in an open connected neighbourhood of \bar{x} . Then, since (h_3) is in force, we can use the strong maximum principle and the Hopf boundary lemma (see for instance Chapter 5 in [25]) to get (3.5), as before.

Remark 3.1. *Note that, in the previous argument, we used assumption (h_3) only for $\bar{t} = 0$ (and only in the case $f(0) = 0$).*

The desired conclusion (3.2) then follows immediately from (3.5) and the C^2 -regularity of u up to the boundary.

From the analysis above, we find the existence of (possible very small)

$$(3.6) \quad \bar{s} = \bar{s}(\bar{\theta}) > 0,$$

such that, for any $0 < s \leq \bar{s}$:

- i) both the triangle $\mathcal{T}_{x_0, s, \bar{\theta}}$ and its reflection w.r.t. $L_{x_0, s, \bar{\theta}}$ are contained in $B_r^+(x_0)$ (as well as their reflections w.r.t. the axis $\{x = x_0\}$),
- ii) both the segment $\{(x_0, y) : 0 \leq y \leq s\}$ and its reflection w.r.t. $L_{x_0, s, \theta}$ are contained in $B_r^+(x_0)$ for every $\theta \in (0, \bar{\theta}]$,
- iii) $u < u_{x_0, s, \bar{\theta}}$ in $\mathcal{T}_{x_0, s, \bar{\theta}}$,
- iv) $u \leq u_{x_0, s, \theta}$ on $\partial(\mathcal{T}_{x_0, s, \theta})$ for every $\theta \in (0, \bar{\theta}]$,
- v) $u < u_{x_0, s, \theta}$ on the set $\{(x_0, y) : 0 < y < s\}$, for every $\theta \in (0, \bar{\theta}]$.

Note that, from iii) – iv), we have that

$$(3.7) \quad \forall s \in (0, \bar{s}), \quad (\mathcal{HT}_{x_0, s, \bar{\theta}}) \quad \text{holds.}$$

To continue the description of our results, we denote by $p := (x, y)$ a general point in the plane and, for a nonnegative solution u of (1.1), we say that u satisfies the property (\mathcal{P}_μ) if *there exists a real number $\mu > 0$ and a point $p \in \{y = \mu\}$ such that $u(p) \neq 0$* .

Equivalently :

$$(\mathcal{P}_\mu) \quad \text{holds if} \quad \{y = \mu\} \cap \{u \neq 0\} \neq \emptyset.$$

For a non trivial u and under the assumptions stated at the beginning of this section we have that the set

$$(3.8) \quad \Lambda^* = \Lambda^*(u) := \{\lambda > 0 : (\mathcal{P}_\mu) \text{ holds for every } 0 < \mu \leq \lambda\}$$

is not empty. The latter claim follows from (3.5) when $f(0) \geq 0$ and from Theorem 6.1 of [16] when $f(0) < 0$ (note that Theorem 6.1 of [16] holds true for functions f which are *only* continuous on $[0, +\infty)$ and so, it applies in our situation since (h_1) is in force).

Therefore we set

$$(3.9) \quad \lambda^* = \lambda^*(u) := \sup \Lambda^* \in (0, +\infty]$$

and also note that, by a continuity argument, if λ^* is finite, we get that $\{y = \lambda^*\} \subseteq \{u = 0\}$.

Next we prove a result that allows to start the moving plane procedure.

Lemma 3.2 (Monotonicity near the boundary). *Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a nonnegative and non trivial solution to (1.1) and assume that f is locally Lipschitz continuous on $[0, +\infty)$.*

Then there exists $\hat{\lambda} > 0$ such that, for any $0 < \lambda \leq \hat{\lambda}$, we have

$$(3.10) \quad u < u_\lambda \quad \text{in} \quad \Sigma_\lambda.$$

Furthermore

$$(3.11) \quad \partial_y u > 0 \quad \text{in} \quad \Sigma_{\hat{\lambda}}.$$

If u is positive, the conclusions above hold when either

i) $f(0) = 0$ and f fulfills (h_1) , (h_2) and (h_3) ,

or

ii) $f(0) \neq 0$ and f fulfills (h_1) , (h_2) and (h_4) .

Proof. Let $\bar{\theta}$ given by (3.3) and $\bar{s} = \bar{s}(\bar{\theta})$ as in (3.6). We showed that, for any $0 < s < \bar{s}$, $(\mathcal{HT}_{x_0, s, \bar{\theta}})$ holds.

We use now Lemma 2.4 as follows: for any fixed $s \in (0, \bar{s})$ and $\theta' \in (0, \bar{\theta})$ we consider the rotation

$$g(t) = (s(t), \theta(t)) := (s, t\theta' + (1-t)\bar{\theta}) \quad t \in [0, 1].$$

Recalling that $(\mathcal{HT}_{x_0, s, \bar{\theta}})$ holds by (3.7), we deduce that also $(\mathcal{HT}_{x_0, s, \theta'})$ holds. Therefore, by the fact that $0 < \theta' < \bar{\theta}$ is arbitrary and by a continuity argument, we pass to the limit for $\theta' \rightarrow 0$ and get

$$u(x, y) \leq u_s(x, y) \text{ in } \Sigma_s \cap \{x \leq x_0\} \text{ for } 0 < s < \bar{s}.$$

The invariance of the considered problem w.r.t. the axis $\{x = x_0\}$ enables us to use the same argument to treat the case of negative θ , yielding

$$u(x, y) \leq u_s(x, y) \text{ in } \Sigma_s \cap \{x \geq x_0\} \text{ for } 0 < s < \bar{s},$$

possibly reducing \bar{s} .

Thus $u(x, y) \leq u_s(x, y)$ in Σ_s for every $s \in (0, \bar{s})$. The desired conclusion (3.10) then follows by taking $\hat{\lambda}$ such that $0 < \hat{\lambda} < \min\{\bar{s}, \frac{\lambda^*}{2}\}$. Here we have used in a crucial way that the property $(\mathcal{P})_\lambda$ holds for every $\lambda \in (0, \hat{\lambda}]$, so that the case $u \equiv u_\lambda$ in Σ_λ is not possible.

Moreover, when f is locally Lipschitz continuous on $(0, +\infty]$, the function $u_\lambda - u > 0$ solves a linear equation of the form $\Delta(u_\lambda - u) = c(x)(u_\lambda - u)$, with c locally bounded on Σ_λ . Therefore, by the Hopf's Lemma, for every $\lambda \in (0, \hat{\lambda}]$ and every $x \in \mathbb{R}$, we get

$$(3.12) \quad -2\partial_y u(x, \lambda) = \frac{\partial(u_\lambda - u)}{\partial y}(x, \lambda) < 0.$$

The latter proves (3.11) when f is locally Lipschitz continuous on $(0, +\infty]$.

If u is everywhere positive, (3.12) is still true since (h_3) (resp. (h_4)) is in force. Indeed, the arguments already used to prove (2.6) and (3.5) and the crucial fact that $u(x, \lambda) > 0$ for every $x \in \mathbb{R}$ lead to

$$(3.13) \quad \begin{cases} \Delta(u_\lambda - u) \leq g(u_\lambda - u) & \text{in } B_x, \\ u_\lambda - u > 0 & \text{in } B_x, \\ u_\lambda(x, \lambda) - u(x, \lambda) = 0 & \forall x \in \mathbb{R}, \end{cases}$$

where $B_x \subset \mathbb{R}^2$ is an open ball centered at (x, λ) . Therefore, since (h_3) (resp. (h_4)) is in force, the boundary lemma gives (3.12). This concludes the proof. \square

Remark 3.3. *Note that, when u is positive, we used only (h_4) .*

Let λ^* be defined as in (3.9). In the case $\lambda^* = \infty$ we set

$$\Lambda = \{\lambda > 0 : u < u_{\lambda'} \text{ in } \Sigma_{\lambda'} \quad \forall \lambda' < \lambda\}.$$

If λ^* is finite we use the same notation but considering values of λ such that $0 < \lambda < \lambda^*/2$, namely

$$\Lambda = \{\lambda < \frac{\lambda^*}{2} : u < u_{\lambda'} \text{ in } \Sigma_{\lambda'} \quad \forall \lambda' < \lambda\}.$$

By Lemma 3.2 we know that Λ is not empty and we can define

$$(3.14) \quad \bar{\lambda} = \sup \Lambda.$$

Now we assume that $\bar{\lambda} < +\infty$, when $\lambda^* = \infty$ (resp. $\bar{\lambda} < \frac{\lambda^*}{2}$, when λ^* is finite) and observe that, arguing as above and under the same assumptions of Lemma 3.2 (cfr. the proof of (3.12)), we deduce that

$$(3.15) \quad u < u_{\bar{\lambda}} \quad \text{on } \Sigma_{\bar{\lambda}},$$

$$(3.16) \quad \partial_y u(x, \lambda) > 0 \quad \forall (x, \lambda) \in \mathbb{R} \times (0, \bar{\lambda}].$$

and then we can prove the following

Lemma 3.4. *Let u and f as in Lemma 3.2. Let λ^* and $\bar{\lambda}$ be as above. Assume that there is a point $x_0 \in \mathbb{R}$ satisfying $u(x_0, 2\bar{\lambda}) > 0$. Then there exists $\bar{\delta} > 0$ such that: for any $-\bar{\delta} \leq \theta \leq \bar{\delta}$ and for any $0 < \lambda \leq \bar{\lambda} + \bar{\delta}$, we have*

$$u(x_0, y) < u_{x_0, \lambda, \theta}(x_0, y),$$

for $0 < y < \lambda$.

Proof. First we note that, by (3.16), we have $\partial_y u(x_0, \bar{\lambda}) > 0$.

We argue now by contradiction. If the lemma were false, we found a sequence of small $\delta_n \rightarrow 0$ and $-\delta_n \leq \theta_n \leq \delta_n$, $0 < \lambda_n \leq \bar{\lambda} + \delta_n$, $0 < y_n < \lambda_n$ with

$$u(x_0, y_n) \geq u_{x_0, \lambda_n, \theta_n}(x_0, y_n).$$

Possibly considering subsequences, we may and do assume that $\lambda_n \rightarrow \tilde{\lambda} \leq \bar{\lambda}$. Also $y_n \rightarrow \tilde{y}$ for some $\tilde{y} \leq \tilde{\lambda}$. Considering the construction of $B_{\tilde{r}}^+(x_0)$ as above and in particular taking into account (3.3) and (3.4), we deduce that $\tilde{\lambda} > 0$ and, by continuity, it follows that $u(x_0, \tilde{y}) \geq u_{\tilde{\lambda}}(x_0, \tilde{y})$. Consequently $y_n \rightarrow \tilde{\lambda} = \tilde{y}$, since we know that $u < u_{\lambda'}$ in $\Sigma_{\lambda'}$ for any $\lambda' \leq \bar{\lambda}$ and we assumed that $u(x_0, 2\bar{\lambda}) > 0$ so that in particular $u(x_0, 0) = 0 < u(x_0, 2\bar{\lambda})$. By the mean value theorem since $u(x_0, y_n) \geq u_{x_0, \lambda_n, \theta_n}(x_0, y_n)$, it follows

$$\frac{\partial u}{\partial V_{\theta_n}}(x_n, y_n) \leq 0$$

at some point $\xi_n \equiv (x_n, y_n)$ lying on the line from (x_0, y_n) to $T_{x_0, \lambda_n, \theta_n}(x_0, y_n)$, recalling that the vector V_{θ_n} is orthogonal to the line $L_{x_0, \lambda_n, \theta_n}$. Since $V_{\theta_n} \rightarrow e_2$ as $\theta_n \rightarrow 0$.

Taking the limit it follows

$$\partial_y u(x_0, \tilde{\lambda}) \leq 0$$

which is impossible by (3.16). □

4. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Since we are assuming that $\bar{\lambda} < +\infty$, when $\lambda^* = \infty$ (resp. $\bar{\lambda} < \frac{\lambda^*}{2}$, when λ^* is finite), by definition of λ^* we can find $x_0 \in \mathbb{R}$ such that $u(x_0, 2\bar{\lambda}) > 0$. Let $B_{\tilde{r}}^+(x_0)$ be constructed as above and pick $\bar{\theta}$ given by (3.3).

Let also $\bar{\delta}$ as in Lemma 3.4. Then fix $\theta_0 > 0$ with $\theta_0 \leq \bar{\delta}$ and $\theta_0 \leq \bar{\theta}$. Let us set

$$s_0 := s_0(\theta_0),$$

such that the triangle $\mathcal{T}_{x_0, s_0, \theta_0}$ and its reflection w.r.t. L_{x_0, s_0, θ_0} is contained in $B_{\tilde{r}}^+(x_0)$ and consequently $(\mathcal{HT}_{x_0, s_0, \theta_0})$ holds. It is convenient to assume that $s_0 \leq \hat{\lambda}$ with $\hat{\lambda}$ as in Lemma 3.2. For any

$$s_0 < s \leq \bar{\lambda} + \bar{\delta}, \quad 0 < \theta < \theta_0,$$

we carry out the *sliding-rotating technique* exploiting Lemma 2.4 with

$$g(t) = (s(t), \theta(t)) := (ts + (1-t)s_0, t\theta + (1-t)\theta_0) \quad t \in [0, 1].$$

By Lemma 3.4 we deduce that the boundary conditions required to apply Lemma 2.4 are fulfilled and therefore, by Lemma 2.4, we get that $(\mathcal{HT}_{x_0, s, \theta})$ holds. We can now argue as in the proof of Lemma 3.2 and deduce that $u(x, y) < u_\lambda(x, y)$ in Σ_λ for any $0 < \lambda \leq \bar{\lambda} + \bar{\delta}$. This provides a contradiction unless $\bar{\lambda} = +\infty$ (resp. $\bar{\lambda} = \frac{\lambda^*}{2}$, if λ^* is finite). Arguing e.g. as in the proof of Lemma 3.2, we deduce

$$\partial_y u > 0 \quad \text{in } \mathbb{R}_+^2 \quad \text{if } \lambda^* = +\infty,$$

while

$$\partial_y u > 0 \quad \text{in } \Sigma_{\frac{\lambda^*}{2}} \quad \text{if } \lambda^* < +\infty.$$

As a consequence of the monotonicity result, we deduce that u is positive in \mathbb{R}_+^2 if $\lambda^* = +\infty$.

Let us now deal with the case when λ^* is finite, that may occur only in the case $f(0) < 0$. We deduce by continuity that

$$u \leq u_{\lambda^*/2} \quad \text{in } \Sigma_{\lambda^*/2}.$$

By the strong comparison principle, we deduce that: either $u < u_{\lambda^*/2}$ or $u \equiv u_{\lambda^*/2}$, in $\Sigma_{\lambda^*/2}$. Note that, by the definition of λ^* , we have that $\{y = \lambda^*\} \subseteq \{u = 0\}$, that also implies $\{y = \lambda^*\} \subseteq \{\nabla u = 0\}$ since u is nonnegative. If $u < u_{\lambda^*/2}$ in $\Sigma_{\lambda^*/2}$, we get by the Hopf's boundary Lemma (see [23]) that $\partial_y(u_{\lambda^*/2} - u) > 0$ on $\{y = 0\}$. Since $\partial_y(u_{\lambda^*/2}) = 0$ on $\{y = 0\}$ (by the fact that $\{y = \lambda^*\} \subseteq \{\nabla u = 0\}$) this provides a contradiction with the fact that u is nonnegative. Therefore it occurs $u \equiv u_{\lambda^*/2}$, in $\Sigma_{\lambda^*/2}$.

Note now that, since $\{y = \lambda^*\} \subseteq \{u = 0\} \cap \{\nabla u = 0\}$, by symmetry we deduce

$$\{y = 0\} \subseteq \{u = 0\} \cap \{\nabla u = 0\}.$$

Therefore we deduce that u is one-dimensional by the *unique continuation principle* (see for instance Theorem 1 of [20] and the references therein). Here we use in a crucial way the fact that f is locally Lipschitz continuous on $[0, +\infty)$. Indeed, for every $t \in \mathbb{R}$, the function $u^t(x, y) := u(x + t, y)$ is a nonnegative solution of (1.1) with $u^t = \nabla u^t = 0$ on $\partial\mathbb{R}_+^2$ and the unique continuation principle implies that $u \equiv u^t$ on \mathbb{R}_+^2 . This immediately gives that u depends only on the variable y , i.e.,

$$u(x, y) = u_0(y) \quad \forall (x, y) \in \mathbb{R}_+^2$$

where $u_0 \in C^2([0, +\infty))$ is the unique solution of $u_0'' + f(u_0) = 0$ with $u_0'(0) = u_0(0) = 0$.

The remaining part of the statement, namely the properties of u_0 , follows by a simple ODE analysis. \square

5. PROOF OF THEOREM 1.2 AND THEOREM 1.3

Proof of Theorem 1.2. Since u is positive we immediately have that $\lambda^* = \infty$. We now observe that, when u is positive, the first part of the proof of Theorem 1.1 holds under the assumptions of Theorem 1.2 (see Lemma 2.4, Lemma 3.2 and Lemma 3.4). Therefore, arguing as in Theorem 1.1, we get

$$\partial_y u > 0 \quad \text{in } \mathbb{R}_+^2.$$

\square

Proof of Theorem 1.3. The proof follows arguing exactly as in the proof of Theorem 1.2, just observing that the translating rotating technique can be performed until we reach the maximal position at the middle of the strip. This provides the fact that u is strictly monotone increasing in Σ_b . To prove that $\frac{\partial u}{\partial y} > 0$ in Σ_b just argue again as in the proof of (3.12) (see also (2.6)). If $u \in C^2(\overline{\Sigma_{2b}})$ and $u = 0$ on $\partial\Sigma_{2b}$, then the technique can be applied in the opposite direction thus proving that u is symmetric about $\{y = b\}$. \square

6. PROOF OF THEOREM 1.4, COROLLARY 1.5 AND THEOREM 1.6

Proof of Theorem 1.4. Since f is locally Lipschitz continuous on $[0, +\infty)$, Theorem 1.1 implies that, either u is one-dimensional and periodic (possibly identically equals to zero) and in this case we are done, or $u > 0$ and $\frac{\partial u}{\partial y} > 0$ everywhere in \mathbb{R}_+^2 . To conclude the proof it remains to consider the second case. First we observe that u is necessarily bounded on \mathbb{R}_+^2 , indeed by Theorem 2.1 of [24] there is a positive constant C , depending only on p, f and the euclidean dimension, such that

$$(6.1) \quad u(x, y) \leq C(1 + \text{dist}^{-\frac{2}{p-1}}((x, y); \partial\mathbb{R}_+^2)) = C(1 + y^{-\frac{2}{p-1}}) \quad \forall (x, y) \in \mathbb{R}_+^2$$

and therefore, the boundedness of u follows by combining the monotonicity of u , i.e., $\frac{\partial u}{\partial y} > 0$ on \mathbb{R}_+^2 , together with the estimate (6.1). Then, by standard elliptic estimates, we also get that $|\nabla u|$ is bounded and so we can apply Theorem 1.6 of our previous work [16] to get that u is one-dimensional. This concludes the proof. \square

Proof of Corollary 1.5. If u is a solution to (1.9), by the strong maximum principle we have either $u \equiv 0$ or $u > 0$. Then, by proceeding as in the proof of Theorem 1.4 we get that either $u \equiv 0$ or u is a positive, bounded, one-dimensional and monotonically increasing function, say $u = u(y)$. The second case is impossible since $\bar{l} := \lim_{y \rightarrow +\infty} u(y)$ would be a positive zero of f , contradicting the assumption (1.7). Therefore, if u is a solution then necessarily $u \equiv 0$ and so $f(0) = 0$. This completes the proof. \square

Proof of Theorem 1.6. By assumption $f \geq 0$ on $[0, +\infty)$ and so, either $u \equiv 0$ or $u > 0$, thanks to the strong maximum principle. Since the linear function $u(x, y) = cy$, $c \geq 0$, is harmonic, to conclude the proof of the first claim it is enough to show that $u > 0 \implies u(x, y) = cy$ for some $c > 0$ (which in turn implies that $f \equiv 0$). To this end we first observe that

$$(6.2) \quad v := \partial_y u > 0 \quad \text{on} \quad \overline{\mathbb{R}_+^2},$$

indeed, $u > 0$ on \mathbb{R}_+^2 implies $v > 0$ on \mathbb{R}_+^2 by Theorem 1.1 and $v > 0$ on $\partial\mathbb{R}_+^2$ by Hopf's Lemma, since $u > 0$ on \mathbb{R}_+^2 and $f(0) \geq 0$ by assumption.

Then we remark that, for every $r > 0$, for every $p \in \overline{\mathbb{R}_+^2}$ and every open ball $B_r(p)$,

$$(6.3) \quad u \in H^3(B_r(p) \cap \mathbb{R}_+^2)$$

by standard elliptic regularity (see, for instance, Theorem 8.13 of [23]) and so we get that

$$(6.4) \quad v \in C^1(\overline{\mathbb{R}_+^2}), \quad v \in H^2(B_r(p) \cap \mathbb{R}_+^2) \quad \forall r > 0, \quad \forall p \in \overline{\mathbb{R}_+^2}.$$

Now, since f is locally Lipschitz continuous, by differentiating the equation satisfied by u and using (6.2) and (6.4) we obtain that v satisfies

$$(6.5) \quad \begin{cases} v > 0 & \text{everywhere in } \overline{\mathbb{R}_+^2}, \\ -\Delta v = f'(u)v \geq 0 & \text{a.e. in } \mathbb{R}_+^2. \end{cases}$$

Then, for any $\psi \in C_c^{0,1}(\mathbb{R}^2)$, we multiply the latter equation by $\psi^2 v^{-1}$ and integrate by parts to get

$$\begin{aligned} 0 \leq - \int_{\mathbb{R}_+^2} \Delta v \psi^2 v^{-1} = \\ - \int_{\mathbb{R}_+^2} \frac{|\nabla v|^2}{v^2} \psi^2 + \int_{\mathbb{R}_+^2} 2v^{-1} \psi \nabla v \nabla \psi + \int_{\partial \mathbb{R}_+^2} \frac{\partial v}{\partial y} v^{-1} \psi^2. \end{aligned}$$

Now we observe that $\frac{\partial v}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \Delta u = -f(0) \leq 0$ on $\partial \mathbb{R}_+^2$, by the assumption (1.10), and therefore we deduce from the latter that

$$(6.6) \quad \int_{\mathbb{R}_+^2} \frac{|\nabla v|^2}{v^2} \psi^2 \leq \int_{\mathbb{R}_+^2} 2v^{-1} \psi \nabla v \nabla \psi$$

and then

$$(6.7) \quad \int_{\mathbb{R}_+^2} \frac{|\nabla v|^2}{v^2} \psi^2 \leq 2 \left[\int_{\mathbb{R}_+^2} \frac{|\nabla v|^2}{v^2} \psi^2 \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}_+^2} |\nabla \psi|^2 \right]^{\frac{1}{2}}$$

which gives

$$(6.8) \quad \int_{\mathbb{R}_+^2} \frac{|\nabla v|^2}{v^2} \psi^2 \leq 4 \int_{\mathbb{R}_+^2} |\nabla \psi|^2.$$

Now, for every $R > 1$, consider the functions $\psi_R \in C_c^{0,1}(\mathbb{R}^2)$ given by

$$\psi_R(x) := \mathbf{1}_{B_{\sqrt{R}}}(x) + \frac{2 \ln(R/|x|)}{\ln R} \mathbf{1}_{B_R \setminus B_{\sqrt{R}}}(x), \quad \forall x \in \mathbb{R}^2$$

and used them into (6.8) to get

$$(6.9) \quad \int_{\mathbb{R}_+^2} \frac{|\nabla v|^2}{v^2} \psi_R^2 \leq 4 \int_{\mathbb{R}_+^2} |\nabla \psi_R|^2 \leq \frac{C}{\log R},$$

where C is a positive constant independent of R . By letting $R \rightarrow +\infty$ in (6.9) we find $\int_{\mathbb{R}_+^2} \frac{|\nabla v|^2}{v^2} = 0$ and so v is a positive constant, say $v \equiv c > 0$. This means that $\partial_y u \equiv c > 0$ and so $u(x, y) = cy$ for every $(x, y) \in \mathbb{R}_+^2$. Therefore $0 = -\Delta(cy) = f(cy)$ for every $y > 0$ and thus $f \equiv 0$, since $c > 0$. This concludes the proof of the first claim.

In view of the discussion above, if $f \not\equiv 0$, the only solution of (1.11) is $u \equiv 0$, which immediately implies item i) and item ii).

□

7. A COUNTEREXAMPLE

In this section we provide a counterexample showing that the conclusion of Theorem 1.2 (and of Theorem 1.3) fails if f satisfies (h_1) , (h_2) but not (h_3) , i.e., the monotonicity property $\frac{\partial u}{\partial y} > 0$ in \mathbb{R}_+^2 , does not hold true if f satisfies (h_1) , (h_2) but not (h_3) . To this end we shall follow section 6 of [13]. With the notation of example 6.3 of [13], the function

$$(7.1) \quad u(x, y) := \begin{cases} u_1(x, y), & y \leq 2, \\ v(x, y - 5), & y > 2, \end{cases}$$

given by formula (6.9) on p. 832 of [13], with $s = 0$ and $x_0 = (0, 5) \in \mathbb{R}^2$, is a smooth entire solution of the equation $-\Delta u = h(u)$ in \mathbb{R}^2 , where h is given by formula (6.8) on p. 832 of [13]. Observe that u is identically zero on the closed affine half-plane $\{(x, y) \in \mathbb{R}^2 : y \leq -1\}$ and positive on the open affine half-plane $\{(x, y) \in \mathbb{R}^2 : y > -1\}$, therefore the function $v(x, y) := u(x, y - 1)$ is a solution of

$$(7.2) \quad \begin{cases} -\Delta v = h(v) & \text{in } \mathbb{R}_+^2, \\ v > 0 & \text{in } \mathbb{R}_+^2, \\ v = 0 & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

which is neither monotone nor one-dimensional. On the other hand h (extended to be equal to the constant $h(2) = 0$ for $t \geq 2$) is a function satisfying (h_1) , (h_2) but not (h_3) . Indeed, h is Hölder continuous on $[0, +\infty)$ and so it satisfies (h_1) . Moreover, h fulfills (h_2) since it is non increasing in a neighbourhood of the points 0, 1 and 2 and smooth on $[0, +\infty) \setminus \{0, 1, 2\}$. Finally, let us prove that h does not satisfy assumption (h_3) at $\bar{t} = 0$. To this end we first observe that $h(t) = -192[t(1 - t^{\frac{1}{4}})]^{\frac{1}{2}}[1 - \frac{5}{4}t^{\frac{1}{4}}]$ for $t \in [0, 1]$ (cf. example 6.1. of [13]) and we suppose, for contradiction, that h satisfies (h_3) at $\bar{t} = 0$. Hence, there exists a function g such that

$$(7.3) \quad h(s) - h(t) \leq g(t - s) \quad \forall s \leq t \in [0, \frac{\delta}{2}]$$

for some $\delta \in (0, 1)$ and fulfilling the integral condition (2.2). By choosing $s = 0$ in (7.3), we have

$$(7.4) \quad -h(t) \leq g(t) \quad \forall t \in [0, \frac{\delta}{2}]$$

and, in view of the explicit form of h near zero, we can find $\eta \in (0, \frac{\delta}{2})$, small enough, such that

$$(7.5) \quad g(t) \geq -h(t) = |h(t)| \geq \gamma t^{\frac{1}{2}} \quad \forall t \in [0, \eta]$$

for some $\gamma > 0$. The latter yields $G(s) \geq \frac{2}{3}\gamma s^{\frac{3}{2}}$ in $[0, \eta]$ and so $\int_0^\eta \frac{ds}{\sqrt{G(s)}} < \infty$, contradicting (2.2). So, assumption (h_3) is not satisfied at $\bar{t} = 0$ (also note that a similar argument shows that h does not satisfy (h_3) neither at 1 nor at 2). Clearly, the same example can be used as a counterexample for Theorem 1.3.

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